

Sweeping out sectional curvature

D. Panov* and A. Petrunin†

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Abstract

We observe that the maximal open set of constant curvature κ in a Riemannian manifold of curvature $\geq \kappa$ or $\leq \kappa$ has a convexity type property, which we call *two-convexity*. This statement is used to prove a number of rigidity statements in comparison geometry.

1 Introduction

Denote by $\mathbb{M}^m[\kappa]$ the model m -space with curvature κ ; i.e., $\mathbb{M}^m[\kappa]$ is the simply connected m -dimensional Riemannian manifold with constant curvature κ . We will also use shortcuts $\mathbb{S}^m = \mathbb{M}^m[1]$ for the unit m -sphere, and $\mathbb{E}^m = \mathbb{M}^m[0]$ for the Euclidean m -space.

In this paper we play with applications of the following lemma. The proof is given in Section 3. This lemma was first discovered by Buyalo in the case of nonpositive curvature; see [3, Lemma 5.8].

1.1. Buyalo's lemma. *Let M be a complete Riemannian manifold with sectional curvature either $\geq \kappa$ or $\leq \kappa$. Let Δ be a tetrahedron in $\mathbb{M}^3[\kappa]$ and Λ be a union of three out of four faces of Δ . Then any immersion $f: \Lambda \rightarrow M$ which is isometric and geodesic on each face can be extended to an isometric geodesic immersion $F: \Delta \rightarrow M$. Moreover, F is uniquely determined by f .*

Here is an immediate corollary:

1.2. Corollary. *Let g be a complete Riemannian metric on \mathbb{R}^3 with curvature ≥ 0 (or ≤ 0) such that all three coordinate planes of \mathbb{R}^3 are flat geodesic hypersurfaces in (\mathbb{R}^3, g) . Then (\mathbb{R}^3, g) is isometric to Euclidean space.*

We would suggest that reader checks that the last statement does not follow from the standard theorems; in particular the splitting theorems can not help here directly.

Let us now introduce some terminology to state further applications.

- ◊ A Riemannian manifold (possibly not complete) of constant curvature κ will be called *κ -flat*.
- ◊ A κ -flat Riemannian manifold (possibly not complete) which satisfies the conclusion of Buyalo's Lemma will be called *two-convex*. This definition is discussed in more details in Section 2.

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- ◇ Given a Riemannian manifold M , its maximal open subset of constant curvature κ will be called κ -flat domain of M and it will be denoted as $\text{Flat}^\kappa M$.

From Buyalo's Lemma one easily gets the following; a formal proof is given in Section 3.

1.3. Observation. *Let M be a complete Riemannian manifold either with curvature $\geq \kappa$ or $\leq \kappa$. Then $\text{Flat}^\kappa M$ is two-convex.*

Here is an application.

1.4. Theorem. *Let $m \geq 3$ and M be a complete connected m -dimensional manifold with curvature ≥ 1 or ≤ 1 which admits a totally geodesic immersion of the closed unit hemisphere $\iota: \mathbb{S}_+^2 \looparrowright M$ and an open neighborhood of $\iota(\mathbb{S}_+^2)$ in M has constant curvature 1. Then M has constant curvature 1.*

Remarks.

- ◇ Note that diameter-sphere rigidity does not help here directly; in principle, the diameter of M might be $< \pi$.
- ◇ Note that \mathbb{CP}^2 equipped with the canonical metric is an example of a space with curvature ≥ 1 and ≤ 4 , which admits totally geodesic immersions of 2-spheres of constant curvature 1 and 4. I.e., the condition in Theorem 1.4 that the curvature is constant in a neighborhood of $\iota(\mathbb{S}_+^2)$ is necessary.
- ◇ In the case of curvature ≥ 1 , Theorem 1.4 also holds in dimension 2; this is proved by Zalgaller in [15]; see Theorem A.2 and the discussion around.

To prove the theorem, one needs to show that if a neighborhood Ω of \mathbb{S}_+^2 in \mathbb{S}^m admits an immersion in a two-convex manifold Φ then Φ has to be complete. Then Observation 1.3 implies that $\text{Flat}^1 M = M$; i.e., M is a spherical space form. In other words, any neighborhood Ω of $\iota(\mathbb{S}_+^2)$ in \mathbb{S}^m is *exhaustive* in the sense of the following definition.

1.5. Definition. *Let Ω be a κ -flat manifold. Assume that any two-convex manifold Φ that appears as the target of an open isometric immersion $\Omega \looparrowright \Phi$ is complete, and at least one such Φ exists. Then we say that Ω is exhaustive.*

Using this definition, we can formulate the following generalization of Theorem 1.4:

1.6. Theorem. *Let M be a complete connected Riemannian manifold with curvature $\geq \kappa$ or $\leq \kappa$. Assume there is an open isometric immersion $\Omega \looparrowright M$ from an exhaustive κ -flat manifold Ω . Then M has constant curvature κ .*

In order to apply this theorem one only has to find a source of exhaustive manifolds. In Section 2, we introduce the notion of the *two-hull* of a κ -flat simply connected manifold Ω ; in some sense this is the minimal simply connected two-convex manifold which contains an *immersed copy* of Ω . It is easy to see that if the two-hull of a manifold Ω is isometric to $\mathbb{M}^m[\kappa]$ then Ω is exhaustive. This permits one to present a number of examples of exhaustive manifolds. This is done in Section 4, here is a list of examples:

(Proposition 4.1.) For $m \geq 3$, any non-empty open subset of $\mathbb{M}^m[\kappa]$ with convex complement.

(Proposition 4.2.) More generally: any open simply connected subset $\Omega \subset \mathbb{M}^m[\kappa]$ which satisfies the following property. For any $p \in \mathbb{M}^m[\kappa]$ there is a 3-dimensional subspace W_p of $\mathbb{M}^m[\kappa]$ containing p (W_p is an isometric copy of $\mathbb{M}^3[\kappa]$) such that $W_p \cap \Omega \neq \emptyset$ and each connected component of $W_p \setminus \Omega$ is a convex set.

In particular,

$$\Omega = \{ (x_1, x_2, \dots, x_m) \in \mathbb{E}^m \mid 1 + x_1^2 + x_2^2 > x_3^2 + x_4^2 + \dots + x_m^2 \}$$

is exhaustive.

(Proposition 4.3.) Any open subset of \mathbb{S}^m which contains the standard 2-dimensional hemisphere. This type of manifolds is used in Theorem 1.4. (This list can be continued.)

Related results.

One outcome of Theorem 1.6 is a sufficient condition on the piece¹ of the model space $\mathbb{M}^m[\kappa]$, which *can not* be exchanged to another piece that has sectional curvature not smaller or not bigger. This condition is nontrivial only for $m \geq 3$.

The similar conditions for scalar and Ricci curvature were studied. The case of deformation with nondecreasing curvature turned out to be very different from the one with nonincreasing curvature.

After rescaling one can only consider three cases $\kappa = -1, 0$ or 1 .

Nondecreasing curvature. If $\kappa = 0$, the case of nondecreasing scalar curvature leads to so called *positive mass conjecture* which is proved by Schoen–Yau and Witten in [13] and [14]. This implies in particular that *the metric of Euclidean space can not be perturbed in a bounded region so that the scalar curvature does not decrease*.

An analogous statement holds for $\kappa = -1$; i.e., *the metric of Lobachevsky space can not be perturbed in a bounded region so that the scalar curvature does not decrease*. The later was proved by Min-Oo in [9].

The case $\kappa = 1$ was considered in [10], where Min-Oo makes an attempt to show that the standard metric on the m -sphere can not be perturbed inside of hemisphere so that the scalar curvature does not decrease. But in [4], Brendle, Marques and Neves find a counterexample. One can not perturb the metric in a sufficiently small domain of sphere, but optimal bounds on such domain seem to be not known.

On the other hand as it was shown by Hang and Wang in [6], one can not perturb the metric of the standard sphere inside its hemisphere with nondecreasing Ricci curvature.

The two-dimensional case of the above statements for $\kappa = 0$ and -1 follows from Gauss–Bonnet formula and the case $\kappa = 1$ was done by Zalgaller (see the Appendix).

Nonincreasing curvature. In [7], Lohkamp proves that for all $m \geq 3$, one can perturb the metric of $\mathbb{M}^m[\kappa]$ in any open region in such a way that its Ricci curvature does not increase. Moreover, this can be done without changing the topology and with arbitrary small change of the geometry of the space.

¹the complement of Ω

In two-dimensional case, attaching a handle can be done in arbitrary small region with decreasing its curvature. On the other hand, if we fix the topology, for $\kappa = 0$ and -1 , Gauss–Bonnet formula prevents any change of metric in bounded regions with nonincreasing curvature. For $\kappa = 1$, even if topology is fixed, the metric can be changed (by inserting a bubble) in arbitrary small open subset so that the curvature in the region decreases. However, it seems that for proper subsets of hemisphere, there is no continuous deformation of this type.

2 Two-convexity and two-hull

2.1. Definition. Let Ω be a m -dimensional κ -flat manifold. We say that Ω is two-convex if the following condition holds: given a tetrahedron² Δ in $\mathbb{M}^3[\kappa]$ with a choice of a subset $\Lambda \subset \Delta$ formed by 3 out of 4 faces, any immersion $f: \Lambda \looparrowright M$ which is isometric and geodesic on each face of Λ can be extended to an isometric geodesic immersion $F: \Delta \looparrowright M$.

2.2. Definition. Let Ω be a simply connected m -dimensional κ -flat manifold. A two-convex manifold Φ is called the two-hull of Ω (briefly $\Phi = \Omega^{(2)}$) if there is an open immersion $\varphi: \Omega \looparrowright \Phi$ such that for any open isometric immersion $\psi: \Omega \looparrowright \Psi$ into a two-convex manifold Ψ there is a isometric immersion $\vartheta: \Phi \looparrowright \Psi$ which makes the following diagram commutative:

$$\begin{array}{ccc} & \Omega & \\ \varphi \swarrow & & \searrow \psi \\ \Phi & \xrightarrow{\vartheta} & \Psi \end{array}$$

Further the immersion $\varphi: \Omega \looparrowright \Phi$ will be called two-hull immersion.

Let us notice that even though for some manifolds Ω the two-hull immersion $\varphi: \Omega \looparrowright \Phi$ is in fact an embedding, in general one should not expect this.

Our next goal is to prove existence of the two-hull.

2.3. Proposition. For any simply connected κ -flat manifold Ω , its two-hull Φ is uniquely defined up to isometry.

Moreover,

- i) If $\varphi: \Omega \looparrowright \Phi$ and $\varphi': \Omega \looparrowright \Phi'$ are two-hull immersions then there is an isometry $\vartheta: \Phi \rightarrow \Phi'$ such that $\varphi' = \vartheta \circ \varphi$.
- ii) Φ is simply connected.

To prove the above proposition, we mimic the proof of existence of ordinary convex hull as the intersection of all convex sets containing the given set.

Proof. Fix a simply connected m -dimensional κ -flat manifold Ω . Note that Ω admits an open isometric immersion $\iota: \Omega \looparrowright \mathbb{M}^m[\kappa]$.

Let us construct a category \mathcal{C}_Ω . The class of objects in \mathcal{C}_Ω is formed by all open isometric immersion $\psi: \Omega \looparrowright \Psi$ where target Ψ is a two-convex κ -flat

²i.e. 3-simplex

manifold, and the morphisms are commutative triangles of isometric immersions

$$\begin{array}{ccc}
 & \Omega & \\
 \psi_1 \swarrow & & \searrow \psi_2 \\
 \Psi_1 & \xrightarrow{\vartheta} & \Psi_2
 \end{array} \tag{*}$$

The category \mathcal{C}_Ω contains at least one object, the immersion $\iota: \Omega \hookrightarrow \mathbb{M}^m[\kappa]$ mentioned above (this is the terminal object of \mathcal{C}_Ω). The existence of the two-hull of Ω is equivalent to the existence of initial object in \mathcal{C}_Ω .

A choice of point $x_\psi \in \Psi$ for each object $\psi: \Omega \hookrightarrow \Psi$ in \mathcal{C}_Ω is called *inverse point system*, if for any morphism as in (*) we have $x_{\psi_2} = \vartheta(x_{\psi_1})$. Note that for any point $p \in \Omega$, the choice of points $x_\psi = \psi(p) \in \Psi$ forms an inverse point system.

Set Φ to be the set of all inverse point systems. Note that Φ comes with natural maps $\varphi: \Omega \rightarrow \Phi$ and $\vartheta_\psi: \Phi \rightarrow \Psi$ for any object $\psi: \Omega \hookrightarrow \Psi$ in \mathcal{C}_Ω such that the following diagram commutes.

$$\begin{array}{ccc}
 & \Omega & \\
 \varphi \swarrow & & \searrow \psi \\
 \Phi & \xrightarrow{\vartheta_\psi} & \Psi
 \end{array}$$

Let us equip Φ with the weakest topology which makes all maps ϑ_ψ continuous. Clearly, with this topology all ϑ_ψ and φ become immersions.

Let Φ' be the maximal open set in Φ which is homeomorphic to m -manifold. Note that Φ' comes with natural κ -flat metric so that each ϑ_ψ is an open isometric immersion of Φ' . It is easy to see that Φ' is two-convex and it contains $\varphi(\Omega)$. Therefore, the isometric immersion $\varphi: \Omega \hookrightarrow \Phi'$ is an object of \mathcal{C}_Ω . Hence there is an immersion $\Phi \hookrightarrow \Phi'$ which commutes with the natural embedding $\Phi' \hookrightarrow \Phi$; i.e., $\Phi = \Phi'$. In other words Φ is isometric to the two-hull of Ω .

The last two statements of the proposition follow easily from above. In particular, Φ coincides with its universal cover $\tilde{\Phi}$ because $\varphi: \Omega \hookrightarrow \Phi$ lifts to $\tilde{\varphi}: \Omega \hookrightarrow \tilde{\Phi}$ ($\pi_1(\Omega) = 0$), and there is a morphism from $\tilde{\varphi}$ to φ in \mathcal{C}_Ω , proving that $\tilde{\varphi}$ and φ are isomorphic because φ is initial in \mathcal{C}_Ω by construction. \square

3 Buyalo's Lemma and the Observation

In this section we prove Buyalo's Lemma 1.1 and Observation 1.3. The proof of the following proposition is left to the reader.

3.1. Proposition. *Let X and Y be (possibly noncomplete) Riemannian manifolds and Γ be an open set of unit speed geodesics in X , covering all points of X . Then $f: X \rightarrow Y$ is an isometric geodesic immersion if and only if for any $\gamma \in \Gamma$, the curve $f \circ \gamma$ is a unit speed geodesic in Y .*

Proof of Buyalo's Lemma. Set $m = \dim M$. Note that the statement of Buyalo's Lemma trivially holds if $m \leq 2$. Further we assume $m \geq 3$.

By choosing an isometric geodesic embedding $\Delta \hookrightarrow \mathbb{M}^m[\kappa]$, we can consider Δ as a subset of $\mathbb{M}^m[\kappa]$. Let us denote by \tilde{p} the common vertex of the faces

in Λ and let $\tilde{x}, \tilde{y}, \tilde{z}$ be the remaining vertexes of Δ . Denote by p, x, y, z the corresponding points in M ; i.e.

$$p = f(\tilde{p}), \quad x = f(\tilde{x}), \quad y = f(\tilde{y}), \quad z = f(\tilde{z}).$$

Set $R = 2 \cdot \text{diam } \Delta$. Assume first that the injectivity radius at any point in $B_R(p) \subset M$ is at least R . In this case f is distance preserving on each face.

3.2. Claim. *$f: \Lambda \rightarrow M$ is a distance preserving map.*

Proof of the claim. On the geodesic $[px]$ consider two unit normal fields that go in the directions of the images of the faces adjacent to $[px]$. Note that both fields are parallel. Thus the angle between the images of the faces in Λ is constant along the common side. Taking the point on geodesic $[px]$ close to p , one can see that angles between faces of $f(\Lambda)$ in M coincide with the corresponding angles in $\Lambda \subset \mathbb{M}^m[\kappa]$.

Consider points

$$\begin{aligned} \tilde{x}' \in [\tilde{p}\tilde{x}], & \quad \tilde{y}' \in [\tilde{p}\tilde{y}], & \quad \tilde{z}' \in [\tilde{p}\tilde{z}], \\ x' = f(\tilde{x}'), & \quad y' = f(\tilde{y}'), & \quad z' = f(\tilde{z}'). \end{aligned}$$

From above, we have that corresponding angles in the triangles $[x'y'z']$ and $[\tilde{x}'\tilde{y}'\tilde{z}']$ are equal; i.e., the angles in triangle $[x'y'z']$ coincide with its comparison angles.

Let \tilde{v} and \tilde{w} be arbitrary points on the sides of triangle $[\tilde{x}'\tilde{y}'\tilde{z}']$ and $v = f(\tilde{v})$ and $w = f(\tilde{w})$. In both cases (curvature $\geq \kappa$ or $\leq \kappa$) the above angle equality implies that

$$|v - w|_M = |\tilde{v} - \tilde{w}|_{\mathbb{M}^m[\kappa]}.$$

(Here $|\cdot - \cdot|_X$ denotes distance function in a metric space X .)

Note that for any $\tilde{v}, \tilde{w} \in \Lambda$ there is a triangle $[\tilde{x}'\tilde{y}'\tilde{z}']$ as above which contains \tilde{v} and \tilde{w} on its sides. Hence the claim follows. \square

Note that there is a map $F: B_R(\tilde{p}) \rightarrow B_R(p)$ satisfying the following properties:

1. $F|_{\Lambda} = f$;
2. $F(\tilde{p}) = p$, and the differential of F at p is an isometry $T_{\tilde{p}} \rightarrow T_p$;
3. F sends all unit speed geodesics through \tilde{p} to unit speed geodesics through p .

3.3. Claim. *The restriction of any such F to Δ satisfies Buyalo's Lemma;*

This claim is proved separately in the two cases:

Proof of the claim in case of curvature $\geq \kappa$. By Toponogov comparison theorem, the diffeomorphism $F: B_R(\tilde{p}) \rightarrow B_R(p)$ is non-expanding. This fact together with Claim 3.2 imply that the restriction of F to Δ is distance preserving on any geodesic in Δ with ends in Λ . Applying Proposition 3.1 we get that the restriction of F to Δ is isometric and geodesic in the interior of Δ and hence the same holds on whole Δ . \square

Proof of the claim in case of curvature $\leq \kappa$. Set Υ to be the set of all minimizing geodesics with ends in $f(\Lambda)$ and let $\tilde{\Upsilon}$ be the subset of M covered by geodesics in Υ . By Toponogov comparison, the diffeomorphism $F: B_R(\tilde{p}) \rightarrow B_R(p)$ is

distance nondecreasing, while its inverse F^{-1} is a distance non-increasing diffeomorphism. Since f is distance preserving, it follows that F^{-1} is isometric on each of the geodesic in Υ ; moreover, any minimizing geodesic between points in Λ can be presented as $F^{-1} \circ \gamma$ for some $\gamma \in \Upsilon$. It follows that $F^{-1}(\tilde{\Upsilon}) = \Delta$, or equivalently $F(\Delta) = \tilde{\Upsilon}$. In particular, F is distance preserving on each minimizing geodesic with ends in Λ . Applying Proposition 3.1 the same way as above, we conclude that the restriction of F to Δ is distance preserving and geodesic. \square

The general case. To treat the general case, choose $\varepsilon > 0$ so that the injectivity radius at any point in $B_R(p)$ is at least $2 \cdot \varepsilon$. Note that one can cover the interior of Δ by an infinite sequence of tetrahedra $\Delta_1, \Delta_2, \dots$ with a choice of three faces Λ_i in each Δ_i such that $\text{diam } \Delta_i < \varepsilon$ and

$$\Lambda_n \subset \Lambda \cup \left(\bigcup_{i < n} \Delta_i \right).$$

for each n . Then it remains to apply the above argument sequentially to $\Delta_1, \Delta_2, \dots$ and pass to the closure. \square

Proof of Observation 1.3. Set $m = \dim M$. Choose any point $p \in \text{Flat}^\kappa M$ and $\tilde{p} \in \mathbb{M}^m[\kappa]$. Choose a map $F: \mathbb{M}^m[\kappa] \rightarrow \text{Flat}^\kappa M$ such that

1. $F(\tilde{p}) = p$, and the differential of F at \tilde{p} is an isometry $T_{\tilde{p}} \rightarrow T_p$;
2. F sends all unit speed geodesics through \tilde{p} to unit speed geodesics through p .

Let $\Omega_p \subset \mathbb{M}^m[\kappa]$ be the maximal open star-shaped w.r.t. \tilde{p} set such that the map F induces an open isometric immersion of Ω_p . Let Ψ_p be the set of all tetrahedra with one vertex at \tilde{p} and three adjacent faces in Ω_p and let $\bar{\Psi}_p$ be the union of all tetrahedra in Ψ_p .

Clearly $\bar{\Psi}_p$ is open and $\bar{\Psi}_p \supset \Omega_p$. According to Buyalo's Lemma, the map F is isometric on each geodesic lying in a tetrahedron from Ψ_p . Applying Proposition 3.1, we get that F is an open isometric immersion $\bar{\Psi}_p \hookrightarrow M$. Thus, $\bar{\Psi}_p = \Omega_p$ for any $p \in \text{Flat}^\kappa M$, hence the result. \square

4 Exhaustive manifolds

Let Ω be a simply connected κ -flat manifold. Denote by $\Omega^{(2)}$ the two-hull of Ω (see Definition 2.2). From the definition of the two-hull, we have that if $\Omega^{(2)}$ is isometric to the model space $\mathbb{M}^m[\kappa]$ then Ω is exhaustive (see Definition 1.5).

In this section we use the above observation to construct examples of exhaustive manifolds. The following two propositions follow directly from the discussion above. (In other words, the proof is left to the reader.)

4.1. Proposition. *Assume $m \geq 3$ and suppose $\Omega \subset \mathbb{M}^m[\kappa]$ is a nonempty open set with convex complement. Then $\Omega^{(2)}$ is isometric to $\mathbb{M}^m[\kappa]$. In particular, Ω is exhaustive.*

Here is a generalization of the above proposition:

4.2. Proposition. *Suppose $m \geq 3$ and suppose $\Omega \subset \mathbb{M}^m[\kappa]$ is a nonempty open set such that through any point $p \in \mathbb{M}^m[\kappa]$ passes a 3-dimensional subspace*

W_p (i.e., an isometric copy of $\mathbb{M}^3[\kappa]$) such that each connected component of $W_p \setminus \Omega$ is a convex set.

Then $\Omega^{(2)}$ is isometric to $\mathbb{M}^m[\kappa]$. In particular, Ω is exhaustive.

The proofs of the following proposition requires some work.

4.3. Proposition. Assume $m \geq 3$ and suppose $\Omega \subset \mathbb{S}^m \stackrel{\text{def}}{=} \mathbb{M}^m[1]$ admits a geodesic isometric immersion $\mathbb{S}_+^2 \hookrightarrow \Omega$. Then $\Omega^{(2)}$ is isometric to \mathbb{S}^m .

Proof. Fix two embeddings $\mathbb{S}_+^2 \hookrightarrow \Omega \hookrightarrow \mathbb{S}^m$, denote their composition by ι . Note that for any point $x \in \mathbb{S}^m \setminus \iota(\partial \mathbb{S}_+^2)$ there is unique embedding $\iota_x: \mathbb{S}_+^2 \hookrightarrow \mathbb{S}^m$ such that $x \in \iota_x(\mathbb{S}_+^2)$ and $\iota_x(z) = \iota(z)$ for any $z \in \partial \mathbb{S}_+^2$. It is easy to see that one can choose a tetrahedron Δ in \mathbb{S}^m , such that one face of Δ belongs to $\iota_x(\mathbb{S}_+^2)$ and contains all points in the set $\iota_x(\mathbb{S}_+^2) \setminus \Omega$, while the rest of the faces is arbitrary close to $\iota(\mathbb{S}_+^2)$, in particular these faces belong to Ω .

Applying to Δ the definition of two-convexity, we get an isometric geodesic immersion of $F: \Delta \rightarrow \Omega^{(2)}$. It is easy to see that the map $x \mapsto F(x)$ is independent on the choice of Δ ; moreover, the obtained map $\mathbb{S}^m \rightarrow \Omega^{(2)}$ is an open isometric immersion. Since $\Omega^{(2)}$ is simply connected (see Proposition 2.3) we have that $\Omega^{(2)}$ is isometric to \mathbb{S}^m . \square

5 Comments and open problems

k -convexity. The definition of two-convexity (2.1) can be generalized to “ k -convexity”; one has to change tetrahedron Δ to a $(k+1)$ -dimensional simplex and Λ to the set formed by k faces out of $(k+1)$ in Δ . In this case, 1-convexity is equivalent to the usual convexity of each connected component of Ω .

In [5, Section $\frac{1}{2}$], Gromov introduced the following closely related notion which we will call further as *Lefschetz- k -convexity*³.

5.1. Definition. An open set Ω in \mathbb{E}^m is *Lefschetz- k -convex* if for any k -dimensional affine subspace A the natural homology homomorphism

$$H_{k-1}(\Omega \cap A) \rightarrow H_{k-1}(\Omega) \quad (**)$$

is injective.

This definition can be generalized to κ -flat manifolds, one only has to replace $\Omega \cap A$ by k -dimensional manifolds Θ that admit proper⁴ isometric geodesic immersion $\Theta \rightarrow \Omega$.

It is easy to show that Lefschetz- k -convexity in \mathbb{E}^m implies our k -convexity. We know that the converse holds in two trivial cases: $k=1$ and $m \leq k+1$, but in all other cases we do not know the answer to the following question.

5.2. Open problem. Is it true that any k -convex open subset of \mathbb{E}^m is Lefschetz- k -convex?

³We state a slight variation of Gromov’s definition; in particular, we restrict our consideration to open sets and change the meaning of k ; in Gromov’s notations Lefschetz- k -convexity in \mathbb{E}^m is called $(m-k)$ -convexity.

⁴An isometric immersion $\iota: \Theta \rightarrow \Omega$ of Riemannian manifolds Θ and Ω is called *proper* if for any point $p \in \Omega$ there is $\varepsilon > 0$ such that each connected component of $\iota^{-1}(\bar{B}_\varepsilon(p)) \subset \Theta$ is compact.

Smooth approximation of two-convex sets. To get a feeling of definition of k -convexity, it is useful to observe the following.

5.3. Proposition. *If Ω is an open subset of \mathbb{E}^m with smooth boundary $\partial\Omega$, then it is k -convex if and only if the hypersurface $\partial\Omega$ has at most $k-1$ negative principle curvatures at any point.*

It is well known that any convex set in \mathbb{E}^m can be approximated by a convex set with smooth boundary. It turns out that for k -convex sets (as well as for Lefschetz- k -convex sets) this is no longer true.

One of the reasons comes from the fact that for k -convex sets with smooth boundary the homeomorphism in $(**)$ is injective for subspaces A of arbitrary dimension; the proof is an exercise in Morse theory, see [5, Section $\frac{1}{2}$]. Thus, any k -convex set which does not satisfy this condition can not be approximated.

To give an explicit example, let $\Omega \subset \mathbb{E}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be the complement to the union of the following two 3-dimensional halfspaces:

$$(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \times \{0\}) \cup (\{0\} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R})$$

Clearly Ω is 2-convex and simply connected, but $H_1(A \cap \Omega) = \mathbb{Z}$ for

$$A = \{(x, y, z, t) \in \mathbb{E}^4 \mid x + y + z + t = 0\}.$$

Two-hull in non simply connected case. The following example shows a problem with extension of the two-hull construction to non simply connected case. Consider an isometric action $\mathbb{Z}_2 \curvearrowright \mathbb{S}^3$ with two fixed points; then take Ω to be the orbit space $\mathbb{S}^3/\mathbb{Z}_2$ with singular orbits removed. Note that Ω admits no open isometric immersion into a two-convex 1-flat manifold. Hence the two-hull of Ω can not be defined in the class of manifolds. On the other hand it can be defined in the class of “Riemannian megafolds”; these creatures were introduced in [12] and under a different name in [8]; they look a lot like Riemannian manifolds, but fail to be topological spaces. (In the above example, the two-hull of Ω is the Riemannian orbifold $(\mathbb{S}^3 : \mathbb{Z}_2)$.)

CAT[κ] spaces and κ -flat domains. A basic question which one might ask is to describe the manifolds (or even simply connected open sets in $\mathbb{M}^m[\kappa]$) which might appear as κ -flat domains in Observation 1.3. In all the examples we know, the following conjecture holds:

5.4. Conjecture. *Let M be a complete connected m -dimensional manifold with curvature $\geq \kappa$ or $\leq \kappa$ and Ω be a connected component of the κ -flat domain of M . Then the completion of the universal cover of Ω is a CAT[κ] space.*

Note if Ω is the interior of a smooth manifold with boundary then for $\kappa \leq 0$ this conjecture follows from Observation 1.3, Proposition 5.3 and the main theorem of Alexander, Berg and Bishop in [1]; for $\kappa > 0$ it only implies that the space is locally CAT[κ]. On the other hand, for general two-convex κ -flat manifolds the conclusion of the Conjecture can fail. This happens for the complement of ε neighborhood of two 2-planes in \mathbb{E}^4 in general position, which is clearly two-convex. (This is yet another way to see that such two-convex set can not be approximated by two convex sets with smooth boundary.)

An analog of the above conjecture for polyhedral spaces is the main topic of our forthcoming paper [11].

More questions. Here is a possible generalization of Proposition 4.3:

5.5. Question. *Is it true that the two-hull of any open simply connected set $\Omega \subset \mathbb{S}^m$ which contains a closed geodesic is isometric to \mathbb{S}^m ?*

The following question of D. Burago and B. Kleiner is open for long time. It is not directly relevant to all above, but it was one of the initial motivations for our work.

5.6. Question. *Is it possible to construct a Riemannian metric g on the product of a torus and an open disc $T^2 \times D^2$ such that the torus $T^2 \times \{0\} \hookrightarrow T^2 \times D^2$ is flat and the curvature is strictly positive outside of $T^2 \times \{0\}$?*

An answer to this question might lead to a better understanding of manifolds with *almost positive curvature* (see [16]).

Let us yet mention two related questions from mathoverflow:

- ◇ Question 55788 about two-convexity and Lefschetz property.
- ◇ Question 50889 about possible generalization of Buyalo's Lemma.

A Zalgaller's rigidity.

Here we briefly repeat the proof of a theorem from [15]. We do this since the result which interests us (Theorem A.2) was not formulated as a separate statement; it appeared as an intermediate statement in the proof.

A.1. Theorem. *Let $A = a_1a_2 \dots a_n$ and $B = b_1b_2 \dots b_n$ be two simple spherical polygons (not necessary convex) with equal corresponding sides. Assume A lies in an open hemisphere and $\angle a_i \geq \angle b_i$ for each i . Then A is congruent to B .*

At first this result might look unrelated to the content of this article. But the proof relies on the following 2-dimensional analog of Theorem 1.4. Recall that *spherical polyhedron* is a simplicial complex equipped with a metric such that each simplex is isometric to a simplex in a standard sphere.

A.2. Theorem. *Let Σ be a spherical polyhedron which is homeomorphic to \mathbb{S}^2 and has curvature ≥ 1 in the sense of Alexandrov. Assume that an open neighborhood of \mathbb{S}_+^2 in \mathbb{S}^2 admits a locally isometric immersion in Σ . Then Σ is isometric to the standard sphere.*

To deduce Theorem A.1 from Theorem A.2, Zalgaller cuts the polygon A from the sphere and glues instead polygon B . As a result he gets the spherical polyhedron Σ as in Theorem A.2. (In fact, if we drop the condition that A lies in a hemisphere, we can obtain this way any spherical polyhedral metric on \mathbb{S}^2 with curvature ≥ 1 .)

Theorem A.2 is proved by induction on the number n of singular points in Σ . The base case $n = 1$ is trivial. To do the induction step, choose two singular points $p, q \in \Sigma$, cut Σ along a geodesic $[pq]$ and patch the hole so that the obtained new polyhedron Σ' has curvature ≥ 1 . The patch is obtained by gluing two copies of a spherical triangle along two sides. For the right choice of the triangle, the points p and q become regular in Σ' and exactly one new singular point appears in the patch. This way, the case with n singular points is reduced to the case with $n - 1$ singular points (if $n > 1$). The patch construction

above was introduced by Alexandrov in his famous proof of convex embeddability of polyhedrons; the earliest reference we have found is [2, VI, §7].

Applying polyhedral approximation, one can extend Theorem A.2 to any surface with curvature ≥ 1 in the sense of Alexandrov; in particular, this shows that Theorem 1.4 holds in addition for $m = 2$ and curvature ≥ 1 .

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